Exact Solutions of Linearized Schwinger-Dyson Equation of Fermion Self-Energy

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Abstract

The Schwinger-Dyson equation of fermion self-energy in the linearization approximation is solved exactly in a theory with gauge and effective four-fermion interactions. Different expressions for the indepedent solutions which respectively submit to irregular and regular ultraviolet boundary condition are derived and expounded.

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I. Introduction

Owing to non-linearity of the Schwinger-Dyson (S-D) equation[1−7] of the fermion self-energy, together with running of the gauge coupling constant being included in, it is scarcely possible to obtain an analytic solution of the equation. As a result, one usually has to solve it by numerical method[8−10]. However, one could still get some analytic solutions of the S-D equation if some linearization approximation to the equation is made. The resulting analytic solutions will be quite useful for discussions of chiral symmetry breaking. In this paper, we will derive and expound the general exact analytic solutions of the S-D equation of fermion self-energy in the linearization approximation.

Consider a theory with vectorial gauge and chirally-invariant effective four-fermion interactions. When running of the gauge coupling constant is taken into account, the S-D equation of the fermion self-energy $\Sigma(p^2)$ in the Landau gauge, after Wick rotation and angular integration, will have the following form[11]:

$$\Sigma(p^2) = m_0(\Lambda) + \frac{3C_2(R)}{16\pi^2}\int_0^{\Lambda^2} dk^2 \frac{k^2\Sigma(k^2)}{k^2+\Sigma^2(k^2)}\frac{\bar{g}^2(\max(p^2,k^2))}{\max(p^2,k^2)} + \frac{hd(R)}{2\pi^2}\int_0^{\Lambda^2} dk^2 \frac{k^2\Sigma(k^2)}{k^2+\Sigma^2(k^2)} \tag{1.1}$$

m_0 is the bare fermion mass, Λ is the ultraviolet (UV) momentum cut-off, $C_2(R)$ is the eigenvalue of the squared Casimir operator of the "color" gauge group in the representation R of the fermion field ψ with the dimension $d(R)$, h is the strength of the chirally-invariant four-fermion interaction $[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]$ and $\bar{g}^2(\max(p^2,k^2))$ is the conventional approximation of the running gauge coupling constant $\bar{g}^2((p-k)^2)$ defined by

$$\bar{g}^2(\max(p^2,k^2)) = \begin{cases} \bar{g}^2(p^2) & \text{if } p^2 > k^2 \\ \bar{g}^2(k^2) & \text{if } k^2 > p^2 \end{cases} \tag{1.2}$$

Set

$$x \equiv p^2, \quad y \equiv k^2 \tag{1.3}$$

Eq.(1.1) will be reduced to that

$$\Sigma(x) = m_0 + \frac{3C_2(R)}{16\pi^2\beta_0}\left[\frac{1}{x\tau(x)}\int_0^x dy\frac{y\Sigma(y)}{y+\Sigma^2(y)} + \int_x^{\Lambda^2} dy \frac{\Sigma(y)}{\tau(y)(y+\Sigma^2(y))}\right] + \frac{hd(R)}{2\pi^2}\int_0^{\Lambda^2} dy\frac{y\Sigma(y)}{y+\Sigma^2(y)} \tag{1.4}$$

In Eq.(1.4) we have used a continuous Ansatz[6] of the running gauge coupling constant

$$\bar{g}^2(q^2) = 1/\beta_0\tau(q^2) \tag{1.5}$$

where

$$\tau(q^2) = \ln(\frac{q^2}{\mu^2} + \xi) \tag{1.6}$$

and

$$\beta_0 = \left[11C_2(G) - \sum_f 4T(R_f)N_f\right]/48\pi^2 \tag{1.7}$$

with the standard denotations in gauge theory. In the flavor sum $\sum_f$ in Eq.(1.7), besides the fermions corresponding to ψ, all the lighter colored fermions will also be included in. The scale parameter μ is optional and the parameter ξ is required to be greater than 1 so as to avoid the infrared (IR) singularity of $\bar{g}^2(q^2)$.

The integral equation (1.4) is equivalent to the following differential equation

$$\omega(x)\Sigma''(x) + [\omega'(x)+1]\Sigma'(x) = -\frac{b}{\tau(x)}\frac{\Sigma(x)}{[x+\Sigma^2(x)]} \tag{1.8}$$

together with the IR boundary condition

$$\Sigma'(0) = -\frac{b}{2(\ln\xi)\Sigma(0)} \tag{1.9}$$

and the UV boundary condition

$$\left\{\left[1 + \frac{a}{b}x\tau(x)\right]\omega(x)\Sigma'(x) + \Sigma(x)\right\}_{x=\Lambda^2} = m_0(\Lambda) \tag{1.10}$$

where

$$b = \frac{3C_2(R)}{16\pi^2\beta_0}, \qquad a = \frac{hd(R)}{2\pi^2} \tag{1.11}$$

and

$$\omega(x) = \left[\frac{1}{x} + \frac{1}{(x+\xi\mu^2)\tau(x)}\right]^{-1} \tag{1.12}$$

We emphasize the following points: 1) The non-linearity of Eq.(1.8) is embodied in the term in the right-handed side of the equation. 2) The IR boundary condition (1.9) comes from the $x \to 0$ limit of the equation

$$\Sigma'(x) = b\frac{d}{dx}\left(\frac{1}{x\tau(x)}\right)\int_0^x dy \frac{y\Sigma(y)}{y+\Sigma^2(y)} \tag{1.13}$$

which is a result of Eq.(1.4). Since Eq.(1.13) is valid for any x, we can define the IR boundary condition by Eq.(1.13) at some non-zero value of x, instead of at $x=0$, if $\Sigma'(x)$ is calculable at that value of x. This fact will have a close bearing on the actual solution of the following linearized S-D equation. 3) The term in Eq.(1.4) relevant to the four-fermion interactions contains no x hence appears only in the UV boundary condition (1.10) rather than in Eq.(1.8) itself.

In Sect.II we will state the linearization approximation for Eq.(1.8) and a necessary change of the IR boundary condition of the solution. In Sect.III the derivation of the general analytic solutions of the linearized S-D equation will be given in detail and in Sect.IV we will conclude with some discussions on forms and features of the independent solutions.

II. The linearization approximation

The non-linear S-D equation (1.4) or (1.8) has no analytic solution. Some analytic solutions could be obtained merely in the linearization approximation of the equation. To see how to make this approximation, we first consider the IR and UV asymptotic solutions of Eq.(1.8).

In the region where x is small, Eq.(1.8) becomes

$$x\Sigma''(x) + 2\Sigma'(x) + \frac{b}{(\ln\xi)}\frac{\Sigma(x)}{[x+\Sigma^2(x)]} = 0 \tag{2.1}$$

Suppose $\Sigma(x)$ has the form

$$\Sigma(x) = x^s(a_0 + a_1 x + a_2 x^2 + \cdots), \qquad \text{when } x \text{ is small} \tag{2.2}$$

and substituting Eq.(2.2) into the IR boundary condition (1.9) i.e.

$$\lim_{x\to 0}\Sigma'(x)\Sigma(x) = -\frac{b}{2\ln\xi} \tag{2.3}$$

we obtain that

$$\lim_{x\to 0}\left\{a_0^2 s x^{2s-1}+(2s+1)a_0a_1x^{2s}+\Big[(s+1)a_1^2+2(s+1)a_0a_2\Big]x^{2s+1}+\cdots\right\}=-\frac{b}{2\ln\xi} \tag{2.4}$$

The only possibility to satisfy Eq.(2.4) with a real a_0 is to set $s=0$. With this result, substituting Eq.(2.2) into Eq.(2.1) we will have

$$2a_1+6a_2x+12a_3x^2+\cdots+\frac{b}{(\ln\xi)a_0^2}\left[a_0-\left(\frac{1}{a_0}+a_1\right)x+\left(\frac{1}{a_0^3}+3\frac{a_1}{a_0^2}+\frac{a_1^2}{a_0}-a_2\right)x^2+\cdots\right]=0 \tag{2.5}$$

Thus we may express $a_1, a_2, a_3\cdots$ by means of $a_0\equiv\Sigma(0)$ and write down the solution of Eq.(2.1) in small x region which is consistent with the IR boundary condition (1.9) as follows:

$$\begin{aligned}\Sigma(x)=\Sigma(0)\Bigg\{1&-\frac{b}{2(\ln\xi)\Sigma^2(0)}x+\frac{b}{6(\ln\xi)\Sigma^4(0)}\left(1-\frac{b}{2\ln\xi}\right)x^2\\&-\frac{b}{12(\ln\xi)\Sigma^6(0)}\left[1-\frac{5b}{3\ln\xi}+\frac{b^2}{3(\ln\xi)^2}\right]x^3+\cdots\Bigg\}\end{aligned} \tag{2.6}$$

where $\Sigma(0)$ is a finite constant.

In the region where $x\to\infty$, we may have two possible assumptions: 1) $\Sigma^2(x)\geq x$ and 2) $\Sigma^2(x)<x$. If $\Sigma^2(x)\geq x$ is assumed, the asymptotic form of Eq.(1.8) will become that

$$x\left(1-\frac{1}{\ln x}\right)\Sigma''(x)+\left(2-\frac{1}{\ln x}\right)\Sigma'(x)=-\frac{b'}{(\ln x)\Sigma(x)},\quad b'=\frac{b}{1+\lambda}\quad\text{with}\quad\lambda\leq 1 \tag{2.7}$$

where the ratio $x/\Sigma^2(x)$ at $x\to\infty$ in the denominator of the right-handed side of Eq.(2.7) has been replaced approximately by the constant λ. This replacement will not change the essential behavior of the asympototic solution. Eq.(2.7) remains to be a non-linear equation. The exact form of its asymptotic solution is in general unknown. However, we can always suppose a typical form of the solution, such as the form of conventional power function, so as to examine whether the assumption that $\Sigma^2(x)\geq x$ when $x\to\infty$ could be consistent with Eq.(2.7) or not. Thus we set that

$$\Sigma(x)\sim C(\ln x)^r x^s \tag{2.8}$$

where C is a constant, then Eq.(2.7) may be reduced to

$$\begin{aligned}C\Big\{-r(r-1)(\ln x)^{r-3}+r(r-2s-1)(\ln x)^{r-2}+[r(2s+1)-s^2](\ln x)^{r-1}+s(s+1)(\ln x)^r\Big\}x^{s-1}\\=-\frac{b'}{C}(\ln x)^{-r-1}x^{-s}\end{aligned} \tag{2.9}$$

When $s\neq 0,-1$, the leading term in the left-handed side is $Cs(s+1)(\ln x)^r x^{s-1}$ whose equality to the right-handed side requires that

$$r=-r-1,\ s-1=-s,\ Cs(s+1)=-b'/C \tag{2.10}$$

and they give that $r=-1/2,\ s=1/2$ and $C^2=-4b'/3$. The results demand $\Sigma^2(x)\sim x/\ln x$, contradictory to the assumption $\Sigma^2(x)\geq x$. When $s=0$ or -1, the leading term in the left-handed side is $C[r(2s+1)-s^2](\ln x)^{r-1}x^{s-1}$ whose equality to the right-handed side requires that

$$r-1=-r-1,\ s-1=-s,\ C[r(2s+1)-s^2]=-b'/C \tag{2.11}$$

and they give that $r = 0,\ s = 1/2$ and $C^2 = 4b'$. The result $s = 1/2$ is obviously opposite to the presupposition $s = 0$ or -1. Therefore, the assumption $\Sigma^2(x) \geq x$ when $x \to \infty$ can not be consistent with Eq.(2.7), at least this is true for the supposed asymptotic form (2.8) of the solution. Alternatively, we may assume that $\Sigma^2(x) < x$ when $x \to \infty$. In this case, Eq.(1.8) approximately becomes that

$$\left(1 - \frac{1}{\ln x}\right) x\Sigma''(x) + \left(2 - \frac{1}{\ln x}\right) \Sigma'(x) = -\frac{b}{(\ln x)} \frac{\Sigma(x)}{x} \tag{2.12}$$

Noting that Eq.(2.12) has now been linearized. We may still substitute the trial solution (2.8) into Eq.(2.12) and obtain the algebraic equation

$$[r(2s+1) - s^2](\ln x)^{r-1} x^{s-1} + s(s+1)(\ln x)^r x^{s-1} = -b(\ln x)^{r-1} x^{s-1} \tag{2.13}$$

where the non-leading terms such as $(\ln x)^{r-2}$, $(\ln x)^{r-3}$ etc. have been neglected. Eq.(2.13) could be satisfied if the term with $(\ln x)^r x^{s-1}$ is removed by setting

$$s(s+1) = 0 \tag{2.14}$$

i.e. $s = 0$ or $s = -1$. Thus we will obtain the UV asymptotic solution of $\Sigma(x)$ with the form

$$\Sigma(x) = A\left(\ln \frac{x}{\mu^2}\right)^{-b} + \frac{B}{x/\mu^2}\left(\ln \frac{x}{\mu^2}\right)^{b-1} \tag{2.15}$$

The fact that the UV asymptotic form (2.12) of Eq.(1.8) has been linearized indicates that the non-linearity of Eq.(1.8) is important only in the IR region. Actually, the linearization approximation of Eq.(1.8) is a good one not only in the UV region but also in the mediate momentum region where x is not very small. The approximation can be made by replacing $\Sigma^2(x)$ in the denorminator of the non-linear term in Eq.(1.8) by $\Sigma^2(0)$ and this will result in the following linearized version of Eq.(1.8)

$$\omega(x)\Sigma''(x) + [\omega'(x) + 1]\Sigma'(x) = -\frac{b}{\tau(x)} \frac{\Sigma(x)}{[x + \Sigma^2(0)]} \tag{2.16}$$

Generally, the $\Sigma^2(0)$ in the right-handed side of Eq.(2.16) may also be replaced by some constant which will be viewed as the average value of $\Sigma(x)$ over small x region. So the linearized Eq.(2.16) will be qualitatively correct for small x where non-linearity is important.

To make it be easy to solve Eq.(2.16) analytically we take two further assumptions. One of them is to suppose that

$$\Sigma^2(0) = \xi\mu^2 \tag{2.17}$$

This is permissive since both the IR parameter ξ and the scale parameter μ are undeterminated theoretically. Of couse, the constraint that $\xi > 1$ will demand that $\Sigma^2(0) > \mu^2$. Another assumption is that the coefficient $\omega(x)$ [Eq.(1.12)] in Eq.(2.16) is replaced approximately by

$$\omega(x) \simeq \frac{\tau(x)}{1 + \tau(x)}(x + \xi\mu^2) \tag{2.18}$$

This approximation is valid if

$$\frac{x}{\xi\mu^2}\left[1 + \frac{1}{\ln\left(\frac{x}{\mu^2} + \xi\right)}\right] \gg 1 \tag{2.19}$$

The condition (2.19) could impose some constraints on ξ. Let $x/\xi\mu^2 \geq r$ then Eq.(2.19) will become that

$$\frac{1}{\ln[(r+1)\xi]} \gg \frac{1-r}{r} \tag{2.20}$$

The inequality (2.20) is obviously valid for $r \geq 1$ and it may be changed into that

$$\xi \ll \frac{1}{r+1} e^{\frac{r}{1-r}} \quad \text{for} \quad 0 < r < 1 \tag{2.21}$$

Hence, in the case with $r < 1$, ξ will have an upper bound. The upper bound will decrease as r goes down. Although for some values of r we could obtain the constraint on ξ which is allowed by phenomenenlonogy (for example, if $r = 3/4$ then the constraint will be $1 < \xi \ll 11.477$),in the following we prefer $r = 1$ to $r < 1$, i.e. we will use the assumption (2.18) continuously down to the scale $x = \xi\mu^2$ without the need to consider any upper bound of ξ.

Under the assumptions (2.17) and (2.18), the linearized equation (2.16) may be reduced to the following form that

$$\frac{\tau(x)}{1+\tau(x)}(x+\xi\mu^2)\Sigma''(x) + \left\{2 - \frac{\tau(x)}{[1+\tau(x)]^2}\right\}\Sigma'(x) = -b\frac{\Sigma(x)}{\tau(x)(x+\xi\mu^2)} \tag{2.22}$$

The solution of Eq.(2.22) must submit to the UV boundary condition (1.10) and some IR boundary condition as well. Since Eq.(2.22) is inapplicable in the region where $x < \xi\mu^2$, we will not be able directly to use the IR boundary condition (1.9) at $x \to 0$. Instead, as was indicated in Sect.I, we may now use Eq.(1.13) and the asymptotic solution (2.6) of $\Sigma(x)$ in small x region to give a new IR boundary condition at $x = \xi\mu^2$, i.e.

$$\Sigma'(x)|_{x=\xi\mu^2} = b\frac{d}{dx}\left(\frac{1}{x\tau(x)}\right)\bigg|_{x=\xi\mu^2} \int_0^{\xi\mu^2} dy \frac{y\Sigma(y)}{y+\Sigma^2(y)} \tag{2.23}$$

where we have also made the assumption that the solution (2.6) of $\Sigma(x)$ in small x region will be approximately extended to $x = \xi\mu^2$. The use of the IR boundary condition (2.23) instead of Eq.(1.9) will allow us to be able both to solve Eq.(2.22) exactly and analytically and to include partially the IR non-linearity of Eq.(2.10) in.

It is noted that the scales of physical chiral symmetry breaking will generally be in the region where $x > \xi\mu^2$, so it is reasonable to use Eq.(2.22) to discuss such kind of problem. The assumptions that the approximation (2.18) and the solution (2.6) are extended to $x = \xi\mu^2$ respectively from above and from below will at most numerically affect the IR boundary condition (2.23) of Eq.(2.22) and not produce an essential impact on physical conclusions.

III. Solutions of the linearized equation

We will find out the exact indepedent solutions of the linearized equation (2.22). Instead of x, let τ to be the new variable, then Eq.(2.22) may be changed into that

$$\frac{\tau}{1+\tau}\Sigma''(\tau) + \left[1 + \frac{1}{(1+\tau)^2}\right]\Sigma'(\tau) + b\frac{\Sigma(\tau)}{\tau} = 0 \tag{3.1}$$

Assume the exact solution of Eq.(3.1) have the form

$$\Sigma(\tau) = e^{s\tau}\tau^r f(\tau) \tag{3.2}$$

where s and r are constants to be determined. Substituting Eq.(3.2) into Eq.(3.1) and dividing each term of the equation by $e^{s\tau}\tau^{r+1}$, we will obtain the differential equation satisfied by $f(\tau)$ as follows:

$$(\tau+1)f''(\tau) + [(2s+1)\tau + 2(r+s+1) + 2(r+1)/\tau]f'(\tau)$$

$$+\{s(s+1)\tau+[2s(r+1)+s^2+r+b]+[r(r+1)+2s(r+1)+2b]/\tau+[r(r+1)+b]/\tau^2\}f(\tau)=0 \tag{3.3}$$

The UV ($\tau \to \infty$) asymptotic solutions of Eq.(3.1) may be obtained from Eq.(3.3) by taking $f(\tau) \equiv$ constant and setting the first and the second term in the coefficient of $f(\tau)$ to be equal to zeroes as well as neglecting all the terms with the orders $1/\tau$ and above. The results are denoted by

$$\Sigma^{\rm UV}_{\rm irreg}(\tau)=\tau^{-b} \quad \text{and} \quad \Sigma^{\rm UV}_{\rm reg}=e^{-\tau}\tau^{b-1} \tag{3.4}$$

Eq.(3.4) is obviously consistent with the UV asymptotic form (2.15) of $\Sigma(x)$. In order to find out the exact solutions of Eq.(3.1) with the UV asymptotic forms (3.4), we must set the first and the fourth term in the coefficient of $f(\tau)$ in Eq.(3.3) to be equal to zeroes, i.e.

$$s(s+1)=0 \tag{3.5}$$

and

$$r(r+1)+b=0 \tag{3.6}$$

Eq.(3.5) will make $\Sigma(x)$ have the correct UV asymptotic form (3.4) and Eq.(3.6), whose left-handed side is just the coefficient of $1/\tau^2$, can also be obtained from the small τ limit of Eq.(3.1) by setting $\Sigma(\tau) \sim \tau^r$, hence it certainly contains the small τ behavier of Eq.(3.1). Since the r given by Eq.(3.6) takes complex values, the corresponding solutions of $\Sigma(\tau)$ will include complex constants. However, a physical solution may be a real linear combination of the two real components of a complex $\Sigma(\tau)$.

Eq.(3.5) gives that $s=0$ and $s=-1$. We will discuss the two cases respectively.

1) s=0. In this case, the solution (3.2) becomes $\Sigma^{(0)}(\tau)=\tau^r f^{(0)}(\tau)$. Then from Eq.(3.3), the equation of $f^{(0)}(\tau)$ can be changed into that

$$(z^2-z)f^{(0)''}(z)-[z^2-\gamma(z-1)]f^{(0)'}(z)-(\alpha z-b)f^{(0)}(z)=0 \tag{3.7}$$

where

$$z=-\tau \tag{3.8}$$

$$\gamma=2(r+1) \tag{3.9}$$

$$\alpha=b+r \tag{3.10}$$

Eq.(3.7) could be solved by means of confluent hypergeometric (Kummer) function[12]. Noting that if $|z| \to \infty$, then Eq.(3.7) becomes

$$zf^{(0)''}(z)+(\gamma-z)f^{(0)'}(z)-\alpha f^{(0)}(z)=0 \tag{3.11}$$

which is just the confluent hypergeometric equation with the solution $f^{(0)}={}_1F_1(\alpha;\gamma;z)$. Hence, the solution of Eq.(3.7) may be written to be that

$$f^{(0)}(z)=a\ {}_1F_1(\alpha;\gamma;z)+g(z) \tag{3.12}$$

where a is a constant. Substituting Eq.(3.12) into Eq.(3.7) and considering that ${}_1F_1(\alpha;\gamma;z)$ obeys Eq.(3.11) we obtain that

$$(z^2-z)g''-[z^2+\gamma(1-z)]g'+(b-\alpha z)g=az\ {}_1F_1{}'(\alpha;\gamma;z)-a(b-\alpha)\ {}_1F_1(\alpha;\gamma;z) \tag{3.13}$$

It is seen that the left-handed sides of both Eq.(3.13) and Eq.(3.7) have the same form. In view of ${}_1F_1(\alpha;\gamma;z)$ has been used in $f^{(0)}(z)$ we may assume that

$$g(z)=cz\ {}_1F_1{}'(\alpha;\gamma;z) \tag{3.14}$$

where c is a constant. Substituting Eq.(3.14) into Eq.(3.13), we will meet the terms with ${}_1F_1''(\alpha;\gamma;z)$ and ${}_1F_1'''(\alpha;\gamma;z)$. However, they can be changed into some combinations of the terms with only ${}_1F_1(\alpha;\gamma;z)$ and ${}_1F_1'(\alpha;\gamma;z)$ by means of Eq.(3.11). In this way, Eq.(3.13) will be reduced to that

$$c\left(\frac{\gamma}{2}-1\right)z\ {}_1F_1'(\alpha;\gamma;z)-c\alpha\ {}_1F_1(\alpha;\gamma;z)=az\ {}_1F_1'(\alpha;\gamma;z)-a(b-\alpha)\ {}_1F_1(\alpha;\gamma;z)$$

which demands that

$$\begin{aligned} a-\left(\frac{\gamma}{2}-1\right)c&=0, \\ -(b-\alpha)a+\alpha c&=0 \end{aligned} \tag{3.15}$$

The a and c have non-zero solution if and only if

$$\alpha-\left(\frac{\gamma}{2}-1\right)(b-\alpha)=0 \tag{3.16}$$

which is obviously satisfied based on Eqs.(3.9), (3.10) and (3.6). As a result, we have $a=\frac{\gamma}{2}-1=r$ if $c=1$ is taken. From this we will have the solution of Eq.(3.7) with the form

$$f^{(0)}(z)=\left(\frac{\gamma}{2}-1\right)\ {}_1F_1(\alpha;\gamma;z)+z\frac{d}{dz}{}_1F_1(\alpha;\gamma;z) \tag{3.17}$$

and the corresponding

$$\Sigma^{(0)}(\tau)=\tau^{r}\left[\left(\frac{\gamma}{2}-1\right)\ {}_1F_1(\alpha;\gamma;-\tau)-\frac{\alpha}{\gamma}\tau\ {}_1F_1(\alpha+1;\gamma+1;-\tau)\right] \tag{3.18}$$

Depending on the UV asymptotic form, the physically-relevant real solutions will have the following two kinds of forms:

$$\left.\begin{array}{c}\Sigma^{(0)}_{\text{irreg}}(\tau)\\ \Sigma^{(0)}_{\text{reg}}(\tau)\end{array}\right\}=\left.\begin{array}{c}A_i^{(0)}\\ A_r^{(0)}\end{array}\right\}\tau^{-\frac{1}{2}-i\eta}\left[\left(\frac{\gamma}{2}-1\right)\ {}_1F_1(\alpha;\gamma;-\tau)-\frac{\alpha}{\gamma}\tau\ {}_1F_1(\alpha+1;\gamma+1;-\tau)\right]+c.c. \tag{3.19}$$

where one of two roots of the algebraic equation (3.6) of r

$$r=-\frac{1}{2}-i\eta,\ \ \eta=\sqrt{b-\frac{1}{4}} \tag{3.20}$$

has been definitely chosen (chosing the other root $r=-\frac{1}{2}+i\eta$ has no difference because the physical solutions depend on only the two real components of $\Sigma^{(0)}(\tau)$). The complex constants $A_i^{(0)}$ and $A_r^{(0)}$ will be determined respectively by the UV asymptotic conditions

$$\Sigma^{(0)}_{\text{irreg}}(\tau)\overset{\tau\to\infty}{\longrightarrow}\Sigma^{\text{UV}}_{\text{irreg}}(\tau)=\tau^{-b} \tag{3.21}$$

and

$$\Sigma^{(0)}_{\text{reg}}(\tau)\overset{\tau\to\infty}{\longrightarrow}\Sigma^{\text{UV}}_{\text{reg}}(\tau)=e^{-\tau}\tau^{b-1} \tag{3.22}$$

By using the asymptotic formula of confluent hypergeometric function[12]

$${}_1F_1(\alpha;\gamma;-\tau)\overset{\tau\to\infty}{\longrightarrow}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^{-\tau}(-\tau)^{\alpha-\gamma}+\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}\tau^{-\alpha} \tag{3.23}$$

we can write

$$\left.\begin{array}{c}\Sigma^{(0)}_{\rm irreg}(\tau)\\ \Sigma^{(0)}_{\rm reg}(\tau)\end{array}\right\} \stackrel{\tau\to\infty}{\longrightarrow} -b\left[\left.\begin{array}{c}A_i^{(0)}\\ A_r^{(0)}\end{array}\right\}\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}+c.c.\right]\tau^{-b}+\left[\left.\begin{array}{c}A_i^{(0)}\\ A_r^{(0)}\end{array}\right\}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}(-1)^{\alpha-\gamma+1}+c.c.\right]e^{-\tau}\tau^{b-1} \tag{3.24}$$

The conditions (3.21) and (3.22) will lead to that

$$A_i^{(0)} = i(-1)^{\alpha}\frac{\pi}{b\sinh(2\pi\eta)|\Gamma(\alpha)|^2}\frac{\Gamma(\alpha)}{\Gamma(\gamma)} \tag{3.25}$$

and

$$A_r^{(0)} = i\frac{\Gamma(\alpha)}{\Gamma(\gamma)}\frac{\sin(\pi\alpha)}{\sinh(2\pi\eta)} \tag{3.26}$$

2) s=-1. In this case the solution (3.2) becomes $\Sigma^{(-1)}(\tau) = e^{-\tau}\tau^r f^{(-1)}(\tau)$ and from Eq.(3.3), $f^{(-1)}(\tau)$ submits to the equation

$$\tau(\tau+1)f^{(-1)''}(\tau) - \left[\tau^2-(\gamma-2)\tau-\gamma\right]f^{(-1)'}(\tau) - (\bar{\alpha}\tau+\gamma-b)f^{(-1)}(\tau) = 0 \tag{3.27}$$

where

$$\bar{\alpha} = 1+r-b \tag{3.28}$$

The solution $f^{(-1)}(\tau)$ can be expressed by ${}_1F_1(\bar{\alpha};\gamma;\tau)$. Assume that

$$f^{(-1)}(\tau) = \bar{a}\ {}_1F_1(\bar{\alpha};\gamma;\tau) + \bar{c}\tau\ {}_1F_1{}'(\bar{\alpha};\gamma;\tau) \tag{3.29}$$

Substituting Eq.(3.29) into Eq.(3.27) and considering that ${}_1F_1(\bar{\alpha};\gamma;\tau)$ satisfies the equation

$$\tau\ {}_1F_1{}''(\bar{\alpha};\gamma;\tau) + (\gamma-\tau)\ {}_1F_1{}'(\bar{\alpha};\gamma;\tau) - \bar{\alpha}\ {}_1F_1(\bar{\alpha};\gamma;\tau) = 0 \tag{3.30}$$

we obtain that

$$[-\bar{a}+(b+\bar{\alpha})\bar{c}]\tau\ {}_1F_1{}'(\bar{\alpha};\gamma;\tau) + [(\bar{\alpha}+b-\gamma)\bar{a}+\bar{\alpha}\bar{c}]\ {}_1F_1(\bar{\alpha};\gamma;\tau) = 0$$

which demands that

$$\begin{array}{rcll} -\bar{a} & + & (b+\bar{\alpha})\bar{c} & = 0\\ (\bar{\alpha}+b-\gamma)\bar{a} & + & \bar{\alpha}\bar{c} & = 0 \end{array} \tag{3.31}$$

The condition on which $\bar{a}$ and $\bar{c}$ have non-zero solution

$$\bar{\alpha} + (b+\bar{\alpha})(b+\bar{\alpha}-\gamma) = 0 \tag{3.32}$$

is certainly satisfied in view of Eqs.(3.28),(3.6) and (3.9). Hence if taking $\bar{c}=1$ then we will have $\bar{a} = b+\bar{\alpha} = r+1 = \gamma/2$ and

$$f^{(-1)}(\tau) = \frac{\gamma}{2}\ {}_1F_1(\bar{\alpha};\gamma;\tau) + \tau\frac{d}{d\tau}\ {}_1F_1(\bar{\alpha};\gamma;\tau) \tag{3.33}$$

and correspondinly,

$$\Sigma^{(-1)}(\tau) = e^{-\tau}\tau^r\left[\frac{\gamma}{2}\ {}_1F_1(\bar{\alpha};\gamma;\tau) + \frac{\bar{\alpha}}{\gamma}\tau\ {}_1F_1(\bar{\alpha}+1;\gamma+1;\tau)\right] \tag{3.34}$$

The real solutions $\Sigma^{(-1)}_{\rm irreg}(\tau)$ and $\Sigma^{(-1)}_{\rm reg}(\tau)$ can be expressed by

$$\left.\begin{array}{r}\Sigma^{(-1)}_{\rm irreg}(\tau)\\ \Sigma^{(-1)}_{\rm reg}(\tau)\end{array}\right\} = \left.\begin{array}{c}A^{(-1)}_i\\ A^{(-1)}_r\end{array}\right\} e^{-\tau}\tau^{-\frac{1}{2}-i\eta}\left[\frac{\gamma}{2}\ {}_1F_1(\bar\alpha;\gamma;\tau)+\frac{\bar\alpha}{\gamma}\tau\ {}_1F_1(\bar\alpha+1;\gamma+1;\tau)\right]+c.c. \tag{3.35}$$

where the complex constants $A^{(-1)}_i$ and $A^{(-1)}_r$ will be determined respectively by the UV asymptotic conditions

$$\Sigma^{(-1)}_{\rm irreg}(\tau) \stackrel{\tau\to\infty}{\longrightarrow} \Sigma^{\rm UV}_{\rm irreg}(\tau)=\tau^{-b} \tag{3.36}$$

and

$$\Sigma^{(-1)}_{\rm reg}(\tau) \stackrel{\tau\to\infty}{\longrightarrow} \Sigma^{\rm UV}_{\rm reg}(\tau)=e^{-\tau}\tau^{b-1} \tag{3.37}$$

By using the asymptotic formula of confluent hypergeometric function

$${}_1F_1(\bar\alpha;\gamma;\tau) \stackrel{\tau\to\infty}{\longrightarrow} \frac{\Gamma(\gamma)}{\Gamma(\bar\alpha)}e^{\tau}\tau^{\bar\alpha-\gamma}+\frac{\Gamma(\gamma)}{\Gamma(\gamma-\bar\alpha)}e^{-i\pi\bar\alpha}\tau^{-\bar\alpha} \tag{3.38}$$

we get from Eqs.(3.36) and (3.37) that

$$A^{(-1)}_i = i(-1)^{\alpha}\frac{\Gamma(\bar\alpha)}{\Gamma(\gamma)}\frac{\sin(\pi\bar\alpha)}{\sinh(2\pi\eta)} \tag{3.39}$$

and

$$A^{(-1)}_r = -i\frac{\pi}{b\ \sinh(2\pi\eta)|\Gamma(\bar\alpha)|^2}\frac{\Gamma(\bar\alpha)}{\Gamma(\gamma)}. \tag{3.40}$$

IV. Some discussions of solutions

We will show that among the four real solutions $\Sigma^{(s)}_{\rm irreg}$ and $\Sigma^{(s)}_{\rm reg}$ $(s=0,-1)$ derived above, only two ones are linearly independent, consistent with the general conclusion that a second order differential equation has two independent solutions.
In fact, $\Sigma^{(0)}(\tau)$ and $\Sigma^{(-1)}(\tau)$ differ by only a constant. Denote

$$\Sigma^{(0)}(\tau)=\tau^{r}f^{(0)}(\tau) \tag{4.1}$$

then $f^{(0)}$ will obey Eq.(3.7) or

$$\tau(\tau+1)f^{(0)''}(\tau)+\left[\tau^2+\gamma(\tau+1)\right]f^{(0)'}(\tau)+(\alpha\tau+b)f^{(0)}(\tau)=0 \tag{4.2}$$

Now write

$$f^{(0)}(\tau)=e^{-\tau}h(\tau) \tag{4.3}$$

then it is easy to verify that $h(\tau)$ obey the equation

$$\tau(\tau+1)h''(\tau)-\left[\tau^2-(\gamma-2)\tau-\gamma\right]h'(\tau)-(\bar\alpha\tau+\gamma-b)h(\tau)=0 \tag{4.4}$$

which has the same form as Eq.(3.27) of $f^{(-1)}(\tau)$. It follows from this that $h(\tau)\propto f^{(-1)}(\tau)$. Hence

$$\Sigma^{(0)}(\tau)=e^{-\tau}\tau^{r}h(\tau)\propto e^{-\tau}\tau^{r}f^{(-1)}(\tau)=\Sigma^{(-1)}(\tau) \tag{4.5}$$

i.e. $\Sigma^{(0)}(\tau)$ and $\Sigma^{(-1)}(\tau)$ differ at most by a constant and both are linearly dependent.

This conclusion may also be reached directly from the explicit expressions of $\Sigma^{(0)}(\tau)$ and $\Sigma^{(-1)}(\tau)$. By means of the property of confluent hypergeometric function[12]

$${}_1F_1(\alpha;\gamma;-\tau)=e^{-\tau}\ {}_1F_1(\gamma-\alpha;\gamma;\tau) \tag{4.6}$$

we can express $\Sigma^{(0)}(\tau)$ in Eq.(3.18) by

$$\Sigma^{(0)}(\tau) = e^{-\tau}\tau^r \left[\left(\frac{\gamma}{2}-1\right) {}_1F_1(\gamma-\alpha;\gamma;\tau) - \frac{\alpha}{\gamma}\tau\ {}_1F_1(\gamma-\alpha;\gamma+1;\tau)\right] \tag{4.7}$$

From Eqs.(3.9), (3.10) and (3.28) we see that

$$\gamma - \alpha = \bar{\alpha} + 1 \tag{4.8}$$

By Eq.(4.8) and the recurrence relation[12]

$$\gamma\ {}_1F_1(\bar{\alpha}+1;\gamma;\tau) = \tau\ {}_1F_1(\bar{\alpha}+1;\gamma+1;\tau) + \gamma\ {}_1F_1(\bar{\alpha};\gamma;\tau) \tag{4.9}$$

we may rewrite Eq.(4.7) in the form that

$$\begin{aligned}
&\Sigma^{(0)}(\tau)\\
&= e^{-\tau}\tau^r\left(\tfrac{\gamma}{2}-1\right)\left[{}_1F_1(\bar{\alpha};\gamma;\tau) + \tfrac{1}{\gamma}\left(1-\tfrac{\alpha}{\gamma/2-1}\right)\tau\ {}_1F_1(\bar{\alpha}+1;\gamma+1;\tau)\right]\\
&= e^{-\tau}\tau^r\tfrac{\gamma-2}{2}\left[{}_1F_1(\bar{\alpha};\gamma;\tau) + \tfrac{2}{\gamma}\tfrac{\bar{\alpha}}{\gamma}\tau\ {}_1F_1(\bar{\alpha}+1;\gamma+1;\tau)\right]\\
&= \tfrac{\gamma-2}{\gamma}e^{-\tau}\tau^r\left[\tfrac{\gamma}{2}\ {}_1F_1(\bar{\alpha};\gamma;\tau) + \tfrac{\bar{\alpha}}{\gamma}\tau\ {}_1F_1(\bar{\alpha}+1;\gamma+1;\tau)\right]\\
&= \tfrac{\gamma-2}{\gamma}\Sigma^{(-1)}(\tau)
\end{aligned} \tag{4.10}$$

where Eqs.(3.6), (3.9),(3.10), (3.28) and the expression (3.34) for $\Sigma^{(-1)}(\tau)$ have been used. Eq.(4.10) shows that $\Sigma^{(0)}(\tau)$ and $\Sigma^{(-1)}(\tau)$ differs by only the constant $(\gamma-2)/\gamma$ indeed. By comparing Eq.(4.7) with Eq.(4.10) and using the relation (4.6) we see that the equality

$$\left(\frac{\gamma}{2}-1\right)\ {}_1F_1(\alpha;\gamma;-\tau) - \frac{\alpha}{\gamma}\tau\ {}_1F_1(\alpha+1;\gamma+1;-\tau) = \frac{\gamma-2}{\gamma}e^{-\tau}\left[\frac{\gamma}{2}\ {}_1F_1(\bar{\alpha};\gamma;\tau) + \frac{\bar{\alpha}}{\gamma}\tau\ {}_1F_1(\bar{\alpha}+1;\gamma+1;\tau)\right] \tag{4.11}$$

is valid. From this equality it may be verified that the real solutions in Eqs.(3.19) and (3.25) satisfy the relations

$$\Sigma^{(0)}_{\text{irreg}}(\tau) = \Sigma^{(-1)}_{\text{irreg}}(\tau) \text{ and } \Sigma^{(0)}_{\text{reg}}(\tau) = \Sigma^{(-1)}_{\text{reg}}(\tau) \tag{4.12}$$

It is emphasized that in the verification of Eq.(4.12), the equality (4.11) hence the relation (4.6) plays a key role. Since the constants $A_i^{(0)}, A_r^{(0)}, A_i^{(-1)}$ and $A_r^{(-1)}$ are determined through the UV asymptotic form of $\Sigma^{(s)}_{\text{irreg}}(\tau)$ and $\Sigma^{(s)}_{\text{reg}}(\tau)$ $(s=0,-1)$ this means that the relation (4.6) must keep to be valid also in the UV asymptotic region. It is just this requirement that determines the selections of the phase factors in the asymptotic expansions (3.23) and (3.38).

In summary, the linearized Schwinger-Dyson equation (3.1) of the fermion self-energy in the theory with running gauge coupling constant has two linearly independent real solutions which can be considered as the the complex solution $\Sigma^{(0)}(\tau)$ or $\Sigma^{(-1)}(\tau)$ or the pairs of real solutions $(\Sigma_{\text{irreg}}(\tau) \equiv \Sigma^{(0)}_{\text{irreg}}(\tau) = \Sigma^{(-1)}_{\text{irreg}}(\tau), \Sigma_{\text{reg}}(\tau) \equiv \Sigma^{(0)}_{\text{reg}}(\tau) = \Sigma^{(-1)}_{\text{reg}}(\tau))$. The physical solutions should be some linear combination of $\Sigma_{\text{irreg}}(\tau)$ and $\Sigma_{\text{reg}}(\tau)$ with the combination coefficients to being determined by the IR and UV boundary conditions (2.23) and (1.10). The application of these physical solutions to problem of chiral symmetry breaking will be discussed elsewhere.

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